STABILITY OF THE CONTRACTED STATE OF A NONEQUILIBRIUM PLASMA

V. I. Artemov and O. A. Sinkevich

UDC 533.951.8

Overheat instability (OI) in a nonequilibrium homogeneous plasma is related to the ionization instability of the electron gas; the energy losses due to collision with heavy particles cannot balance the fluctuations in the Joule heat generation. The criterion for the development of OI in a nonequilibrium plasma is the condition $d \ln \tau_e / d \ln T_e > 1/2$; τ_e is the time of the loss of momentum when an electron collides with heavier particles [1, 2]. Overheat instability also develops when the energy losses from the electron gas are determined by radiation [3-6], heat exchange with the walls [4, 7], or inelastic losses [2].

When OI develops nonlinearly, it leads to the formation of structures with inhomogeneous distributions of current and electron temperature, which constitute layers in the plane case or columns in the cylindrical case with increased density of current and electron temperature. The theory of such structures in semiconductors – narrow and wide domains – has been constructed in [8, 10]. Analogous structures (contracted state of the discharge) in a gas-discharge plasma have been investigated in [4, 7, 11-13]. In the contracted state the discharge occupies a limited region, and we observe a regime with a normal current density; the current density is independent of the total current. The experiments in [14] confirm the theoretical conclusions. The development of OI in equilibrium plasma [15, 16] may also lead to the appearance of nonstationary structures [17].

In order to explain a number of experimental facts, we much clarify the question of the stability of an inhomogeneous current distribution, and in the case of instability we must also clarify the question of the characteristic time of its development – the time during which the inhomogeneous state exists. The stability of an inhomogeneous current distribution has been investigated in [8-10, 18, 19], in which it was shown that a plane narrow domain for which the width of the column is of the same order as the thickness of the column wall will be unstable, and a wide plane domain with a column width much greater than the wall thickness will be stable for a sufficiently high resistance of the external circuit (a given current regime). Instability of the plane narrow domain leads to its subdivision into cylindrical narrow domains which are stable [8].

The investigations of [8] are restricted to perturbations which are homogeneous in the direction of the principal current $[\mathbf{k}\cdot\mathbf{j}_0 = 0$, ($\mathbf{k}$ is the wave vector of the perturbation, $\mathbf{j}_0$ is the vector of the unperturbed current with boundary conditions $\partial T_e'/\partial\eta^0 = 0$, T_e' is the normal to the boundary of the specimen)].

In the present study we consider the construction of inhomogeneous distributions of current and temperature, when the energy losses from the electron gas are determined by elastic collisions with heavy particles, and we investigate the stability of these current distributions with respect to three-dimensional perturbations. Unlike [4, 8], we consider a broader class of boundary conditions for the electron temperature $L(S, \nabla S, \mathbf{E}) = 0$ (in [4, 8] the condition $\partial T_e/\partial\eta = 0$ at the wall was investigated). In order to construct solutions and investigate the stability, we use the method of singular expansions [20].

1. We assume that the electron concentration n and the electron temperature T are connected by Saha's equation. We consider a strongly nonequilibrium plasma ($T \gg T_a$), in which there is an admixture of a readily ionizable component. In this case the system of equations of the medium [21] in dimensionless form reduces to the following:

$$C\partial S/\partial t + \Lambda U \nabla\Phi\nabla S = \Lambda^2\nabla^2 S + \sigma(\nabla\Phi)^2 - F_-,$$
$$\nabla^2\Phi + (d\ln\sigma/dS)\nabla\Phi\nabla S = 0, \tag{1.1}$$

where $U = T^{-1}(a_T - 3/2)$; $C = (\tau T)^{-1}\ [3/2\cdot\varepsilon + a_T(T^{-1} + 3/2\cdot\varepsilon)]$; $a_T = \partial\ln n/\partial\ln T$; $F_- = nT\tau^{-1}$; λ, σ are the dimensionless electronic thermal conductivity and electrical conductivity, respectively ($\lambda = \sigma T$); τ, dimen-

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 15-26, September-October, 1979. Original article submitted September 25, 1978.

0021-8944/79/2005-0533$07.50

sionless time of loss of momentum; S, heat-conduction potential ($dS = \lambda dT$); Φ, generalized electric field potential, related to the electric field **E** by the formula

$$\nabla\Phi = \mathbf{E} + \Lambda(1 + a_T)\nabla T.$$

As the linear scale in the system (1.1), we have taken the characteristic dimension of the region, b, and as the time scale we have taken $In^*\sigma^*/(\sigma^* E^*)^2$. If the thermal conductivity of the electrons satisfies the Wiedemann–Franz law, then

$$\Lambda = k\sigma^* T^*/(ej^* b), \; \varepsilon = kT^*/I,$$

where I is the ionization potential and the rest of the notation is the same as in [20].

We shall try to find the solution of the system (1.1) in the region $R = \{-1 \leqslant x \leqslant 1, -l_y \leqslant y \leqslant l_y\}$ in the plane (x, y), part of whose boundary $\overline{R}_1 = \{y = \pm l_y\}$ consists of ideally conductive electrodes, while another part $\overline{R}_2 = \{x = \pm 1\}$ consists of insulators with the boundary conditions ($\boldsymbol{\eta}^0$ is a vector normal to the insulator, $\boldsymbol{\chi}^0$ is a vector lying in the plane of the electrode):

$$\partial\Phi/\partial\boldsymbol{\chi}^0 = \Lambda(1 + a_T)\partial T/\partial\boldsymbol{\chi}^0 \text{ on } \overline{R}_1, \; \partial\Phi/\partial\boldsymbol{\eta}^0 = 0 \text{ on } \overline{R}_2,$$
$$S = S_b(\overline{R}), \; \overline{R} = \overline{R}_1 \cup \overline{R}_2. \quad (1.2)$$

In all of our subsequent discussion, we shall not specify the form of the boundary condition for the heat-conduction potential. In constructing a stationary solution in the case $\Lambda \ll 1$ (which is true for most problems), we can use the method proposed in [20]. The solution of the problem is represented in the form of two expansions in the small parameter Λ: the exterior expansion

$$T = \sum_{k=0}^{\infty} \Lambda^k T_k(x, y), \; \Phi = \sum_{k=0}^{\infty} \Lambda^k \Phi_k(x, y) \quad (1.3)$$

and the interior expansion

$$T = \sum_{k=0}^{\infty} \Lambda^k \Theta_k(x^*, y^*), \; \mathbf{E} = \sum_{k=0}^{\infty} \Lambda^k \mathbf{E}_k(x^*, y^*),$$

where $x^* = x/\Lambda$, $y = y^*$ or $x^* = x$, $y = y/\Lambda$.

In constructing plane stationary solutions of the equations of the interior expansion in a closed region, we can have solutions of two types: solutions of the boundary-layer type at the boundary $\overline{R}$ and solutions of the standing-ionization-wave type, situated inside the region R. For the existence of the latter, it is necessary that the function

$$F(S) = \sigma(S)E^2 - F_-(S)$$

have three zeros. This condition is satisfied if

$$a_\tau \equiv d \ln \tau / d \ln T > 1/2.$$

The first zero is connected with the ionization of the admixture, the second with the effect of the Coulomb collisions, and the third with the ionization of the main gas. The construction of the zeroth approximation of the interior expansion is discussed in [20], and therefore we shall not deal with the question here.

The equations of the exterior expansion can be obtained by substituting (1.3) into (1.1) and collecting terms of the same degree in Λ. In the zeroth approximation the system (1.1) in the stationary case reduces to a quasilinear equation in the function Φ_0. We use only electrodynamic boundary conditions. The particular solution of this system is a homogeneous solution (E_{x0} = const, E_{y0} = const), which is used for constructing the solution. In this case the heat-conduction potential is also homogeneous and is found from the solution $F(S_0) = 0$. The stationary solution is constructed by joining the exterior and interior expansions. Since the region of variation of the heat-conduction potential in the ionization wave and in the boundary layer $\sim \Lambda$, as $\Lambda \to 0$, the boundary layer and the ionization wave may be regarded as discontinuities, and therefore solutions with ionization waves will be called discontinuous.

Depending on the parameters of the external circuit, two types of stationary solutions – continuous and discontinuous – may arise. The former are homogeneous solutions (S_0 = const, E_{y0} = const, $E_{x0} = 0$) in almost the entire region R (the exterior domain of solution), except for narrow regions of thickness $\sim \Lambda$ at the boundary (the boundary-layer region), where the potential S_0 changes sharply from the value in the exterior region to the boundary value S_0 (an example of such a potential distribution is given in Fig. 1a). The volt–ampere characteristic (VAC) of such a solution is shown by the dashed curves in Fig. 2; here

$$\langle j_{0y}\rangle = \int_{-1}^{1} j_{0y}dx\,;$$

the VAC is not single-valued over the field E_{y0}, since $S_0(E_{y0})$ is not a single-value function (in the segment ef (Fig. 2) $d\ln\tau/d\ln T > 1/2$).

The discontinuous solutions consist of one or more stationary layered waves. A layered wave is a combination of two ionization waves (wide domains in the terminology of [8]) or stationary solitions (narrow domains) in the exterior region. The plane of the waves is parallel to the insulators. A layered wave consists of three homogeneous solutions (S_{01}, S_{03}, S_{01}, Fig. 1b), connected to each other by two narrow ($\sim \Lambda$) zones of sharp variation of the potential from S_{01} to S_{03} and from S_{03} to S_{01} (ionization waves). On the VAC the segment bc corresponds to the layered wave and the segment eb to the solitions. Since the continuous structure of the ionization wave exists only for a completely determined field E_c, the VAC of the layered wave is a straight line parallel to the $\langle j_{0y}\rangle$ axis. The value of the field E_c can be determined from the condition [4]

$$\int_{S_{01}(E_c)}^{S_{03}(E_c)} F(S,\, E_c)\, dS = 0.$$

The structure of the stationary solutions depends substantially on the type of boundary conditions for S on the insulator.

As was shown in [20], to the distribution of the potential S in the boundary layer there corresponds in the phase plane (dS/dx^*, S) a phase trajectory starting from the singular point $(0, S_0)$ and extending to $S = S_b$. We consider the most real case $S_b < S_0$. Since the singular point $(0, S_0)$, for values of S_0 satisfying the inequality

$$a_{\tau 0} > 1/2$$

is a center, in this case there is no stationary structure of the boundary layer on the insulators. Therefore, the segment ef on the VAC (see Fig. 2) is not realized. For values of S_0 corresponding to the segment fc, the phase trajectory starting from $(0, S_0)$ cannot reach the value $S = S_b$, and therefore this segment of the VAC is not realized either. Taking account of the slow variation of the parameters of heavy particles can lead to an evolution of the resulting stationary structures for electron and electrodynamic quantities.

2. We consider the stability of one-dimensional distributions of the current (electric field) and temperature (this corresponds to the case $l_y \to \infty$). Suppose that the channel is unbounded in the direction of the y and z axes but is bounded in the direction of the x axis ($x = \pm 1$) by insulators, and the unperturbed field extends in the direction of the y axis. If the current and temperature distributions are uniform in the exterior region, then, using the method proposed in [22] and representing the variation of the perturbations as func-

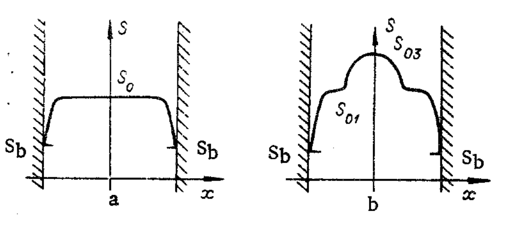

Fig. 1

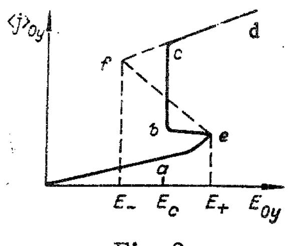

Fig. 2

tions of time and the coordinates x and z in the form $\sim \exp(ik_y y + ik_z z - \omega t)$, we can obtain from the dispersion equation (as the field scale we took the unperturbed field, and as the characteristic length we took the half-width of the channel) the following relation:

$$\operatorname{Re}\omega = \frac{1}{C_0\left[1+\left(\frac{2k}{\pi m}\right)^2\right]}\left\{1 - 2a_{\tau 0} + 2\Lambda^2 k^2 + \left(\frac{2k}{\pi m}\right)^2\left[2\left(a_{T0} + a_{\tau 0}\right) + \frac{k^2}{k_y^2}\left(1 - 2a_{\tau 0} + \Lambda^2 k^2\right)\right]\right\},$$
$$\operatorname{Im}\omega = \Lambda k_y\left(a_{T0} - 3/2\right),\ k = \sqrt{k_y^2 + k_z^2},\ m = 1, 2, \ldots \tag{2.1}$$

Here C_0, a_{T0}, $a_{\tau 0}$ are the values of the functions defined in Eq. (1.1), taken for unperturbed parameters. In obtaining (2.1), we disregarded the boundary layer, and the boundary conditions were given on the boundary of the boundary layer, since for $\Lambda \ll 1$ the dispersion relation depends weakly on the type of boundary conditions for the perturbations on the insulator [22].

If $\partial S_0/\partial x = 0$ for $x = \pm 1$, then the homogeneous states with $a_{\tau 0} > 1/2$ are unstable, and the perturbations with $k = 0$ develop most rapidly. If $\partial S_0/\partial x \neq 0$ for $x = \pm 1$, then the homogeneous states are stable, since the states with $a_{\tau 0} > 1/2$ are not realized.

For nonuniform current and temperature distributions in the exterior region (for discontinuous solutions) we write (for simplicity, $C_0 = \text{const}$):

$$S = S_0(\xi) + \Theta(\xi)\exp\left(ik_y y + ik_z z - \frac{\omega}{C_0}t\right),$$
$$\Phi = E_{y0}y + \Lambda\Psi(\xi)\exp\left(ik_y y + ik_z z - \frac{\omega}{C_0}t\right), \tag{2.2}$$

where $\xi = x/\Lambda$; the subscript refers to the unperturbed solution; (Θ and $\Lambda\Psi$ are quantities of the same order). Substituting (2.2) into (1.1) and disregarding the nonlinear terms, we reduce the investigation of the stability of the inhomogeneous state to the following eigenvalue problem:

$$H_1\Psi = -i\Lambda k_y E_{y0}\frac{d\ln\sigma_0}{dS_0}\Theta,$$
$$(H_0 + \omega)\Theta = H_2\Psi + \Lambda^2 k^2\Theta + i\Lambda k_y E_{y0}U_0\Theta, \tag{2.3}$$

where $H_1 = \frac{1}{\sigma_0(\xi)}\frac{d}{d\xi}\sigma_0(\xi)\frac{d}{d\xi} - \Lambda^2 k^2$; $H_0 = \frac{d^2}{d\xi^2} - V(\xi)$;

$$H_2 = U_0\frac{dS_0}{d\xi}\frac{d}{d\xi} - 2i\Lambda k_y\sigma_0(\xi)E_{y0};$$
$$V(\xi) = -\frac{d}{dS_0}\left(\sigma_0 E_{y0}^2 - F_{-0}\right);$$

$k = \sqrt{k_y^2 + k_z^2}$, with boundary conditions:

$$\xi = \pm\Lambda^{-1},\ \frac{d\Psi}{d\xi} = 0,\ a_1\frac{d\Theta}{d\xi} + ik_y a_3\Lambda\Psi + a_2\Theta = 0. \tag{2.4}$$

Here the constants a_1, a_2, a_3 are connected with the operator L by the relations

$$a_1 = \left[\frac{\partial L}{\partial(\nabla S)}\right]_{\nabla\Phi, S},\ a_2 = \left[\frac{\partial L}{\partial S}\right]_{\nabla S, \nabla\Phi},\ a_3 = \left[\frac{\partial L}{\partial(\nabla\Phi)}\right]_{\nabla S, S}.$$

We consider the problem of the stability of a nonuniform current distribution in the case $k_y \sim k_z \sim 1$, $a_3 = 0, \Lambda \ll 1$. If we introduce Green's function of the equation

$$H_1 G(\xi, \eta) = \delta(\xi, \eta),\ \left.\frac{\partial G}{\partial\xi}\right|_{\xi=\pm\Lambda^{-1}} = 0$$

(here $\delta(\xi - \eta)$ is the Dirac delta function), system (2.3) can be reduced to a single integrodifferential equation for the function Θ:

$$(H_0+\omega)\Theta = -H_2\left[i\Lambda k_y E_{y0}\int_{-\Lambda^{-1}}^{\Lambda^{-1}} G(\xi,\eta)\frac{d\ln\sigma_0}{\partial S_0}\Theta(\eta)\,d\eta\right]+i\Lambda k_y E_{y0}U_0\Theta+\Lambda^2k^2\Theta,$$
$$\xi=\pm\Lambda^{-1},\ a_1\frac{\partial\Theta}{\partial\xi}+a_2\Theta=0. \tag{2.5}$$

Since the system $H_1\Psi=0$, $d\Psi/d\xi(\pm\Lambda^{-1})=0$ has only a trivial solution, Green's function always exists and is unique [23]. Green's function satisfies the following conditions:

$$G(\xi,\eta,k)<0,\ -\Lambda^{-1}\leqslant\xi,\eta\leqslant\Lambda^{-1},$$
$$G\sim O(\Lambda^{-1}),\ \left(\frac{\partial G}{\partial\xi}\right)_\eta\sim O(1)$$

and, to within terms which are $\sim\Lambda$, has the form

$$G(\xi,\eta,k)=-\frac{1}{\Lambda k}\frac{\operatorname{ch}(k\Lambda\eta-k)\operatorname{ch}(k\Lambda\xi+k)}{\operatorname{sh}(2k)},\ \xi<\eta. \tag{2.6}$$

For $\xi>\eta$ Green's function is obtained from (2.6) by interchanging ξ and η.

The original system (2.3) has a singularity with respect to the parameter Λ (the singularity arises from the presence of the Λ^{-1} term in the boundary conditions (2.4)). It is possible to avoid this singularity in Eq. (2.5) by representing

$$\Theta=\sum_{m=0}^{\infty}\Lambda^m\Theta^{(m)}(\xi,\Lambda),\ \omega=\sum_{m=0}^{\infty}\Lambda^m\omega^{(m)}(\Lambda), \tag{2.7}$$

where $\Theta^{(m)}$ and $\omega^{(m)}$ are of formal order unity with respect to Λ. The expansion (2.7) is not asymptotic in the usual sense (since the functions $\Theta^{(m)}$ depend on the parameter Λ) and have the form of a composite expansion in the parameter Λ [24].

For the zeroth terms in expansion (2.7) we obtain the equation

$$(H_0+\omega^{(0)})\Theta^{(0)}=0 \tag{2.8}$$

with the boundary conditions

$$\xi=\pm\Lambda^{-1},\ a_1\frac{d\Theta^{(0)}}{d\xi}+a_2\Theta^{(0)}=0.$$

Equation (2.8) is analogous to the Schrödinger equation for a particle with potential $V(\xi)$. The function $V(\xi)$ consists of one potential well for the ionization (recombination) wave and two potential wells at a distance $\sim\Lambda^{-1}$ from each other for the layered wave (Fig. 3a, solid curve), and at a distance ~ 1 for a soliton (Fig. 3b, solid curve). The function $V(\xi)$ was constructed without taking account of the boundary layer, whose effect on stability will be discussed below. Since the operator H_0 is an Hermitian operator, the eigenfunctions $\Theta_m^{(0)}$ ($m = 0, 1, 2, \ldots$) form a complete orthonormal system

$$\int_{-\Lambda^{-1}}^{\Lambda^{-1}}\Theta_m^{(0)}\Theta_l^{(0)}d\xi=\delta_{lm} \tag{2.9}$$

(if C_0 depends on ξ, then in (2.9) the function $C_0(\xi)$ appears under the integral sign), and the eigenvalues $\omega_0^{(0)}<\omega_1^{(0)}<\ldots$ are real [22]. The equation for the first approximation can be represented in the form ($\Theta_m^{(1)}$ is a correction $\sim\Lambda$ for the function $\Theta_m^{(0)}$)

$$(H_0+\omega_m^{(0)})\Theta_m^{(1)}=-ik_yE_{y0}U_0\frac{dS_0}{d\xi}\int_{-\Lambda^{-1}}^{\Lambda^{-1}}\frac{\partial G}{\partial\xi}\frac{d\ln\sigma_0}{dS_0}\Theta_m^{(0)}d\eta$$
$$-\Lambda^2k_y^2E_{y0}\sigma_0\int_{-\Lambda^{-1}}^{\Lambda^{-1}}G(\xi,\eta)\frac{d\ln\sigma_0}{dS_0}\Theta_m^{(0)}d\eta+ik_yE_{y0}U_0\Theta_m^{(0)}-\omega_m^{(1)}\Theta_m^{(0)} \tag{2.10}$$

with the boundary conditions

$$\xi=\pm\Lambda^{-1},\ a_1\frac{d\Theta_m^{(1)}}{d\xi}+a_2\Theta_m^{(1)}=0.$$

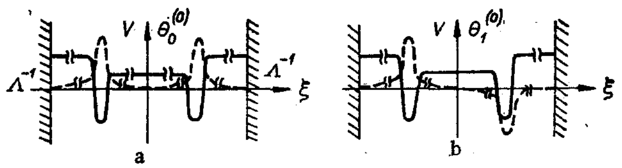

Fig. 3

Expanding $\Theta_m^{(1)}$ in a series in $\Theta_l^{(1)}$

$$\Theta_m^{(1)} = \sum_{l=0}^{\infty} A_{ml}\Theta_l^{(0)}, \tag{2.11}$$

we can determine the quantities $\omega_m^{(1)}$ and A_{ml} $(m \neq l)$:

$$\omega_m^{(1)} = -2k_y^2E_{y0}^2\alpha_{mm} + ik_yE_{y0}\beta_{mm},$$
$$A_{ml} = \frac{1}{\omega_l^{(0)} - \omega_m^{(0)}}\left[2k_y^2E_{y0}^2\alpha_{ml} - ik_yE_{y0}\beta_{ml}\right],$$

where

$$\alpha_{ml} = \Lambda \int_{-\Lambda^{-1}}^{\Lambda^{-1}} \sigma_0(\xi)\,\Theta_l^{(0)}(\xi)\left[\int_{-\Lambda^{-1}}^{\Lambda^{-1}} G(\xi, \eta)\left.\frac{d\ln\sigma_0}{dS_0}\right|_{S_0(\eta)} \Theta_m^{(0)}(\eta)\,d\eta\right]d\xi;$$

$$\beta_{ml} = \int_{-\Lambda^{-1}}^{\Lambda^{-1}} U_0(\xi)\,\Theta_l^{(0)}(\xi)\left[\Theta_m^{(0)}(\xi) - \frac{dS_0}{d\xi}\int_{-\Lambda^{-1}}^{\Lambda^{-1}} \frac{\partial G}{\partial\xi}\left.\frac{d\ln\sigma_0}{dS_0}\right|_{S_0(\eta)} \Theta_m^{(0)}(\eta)\,d\eta\right]d\xi.$$

Continuing with this procedure, we can construct all the terms of the expansion (2.7). For the real component of ω we have

$$\operatorname{Re}\omega = \omega_m^{(0)} - 2\Lambda k_y^2E_{y0}^2\alpha_{mm}. \tag{2.12}$$

Since $G < 0$, $\alpha_{mm} < 0$ (for levels lying in the potential well) and the question of stability reduces to the construction of the spectrum of the operator H_0. For $k_y = 0$, Eq. (2.5) becomes the following:

$$(H_0 + \omega - \Lambda^2k_z^2)\Theta = 0. \tag{2.13}$$

Equation (2.13) can be investigated without using expansions in the parameter Λ, in a manner analogous with [8].

By direct differentiation, we can convince ourselves that the function $dS_0/d\xi$ with the boundary conditions $dS_0/d\xi = 0$ for $x = \pm 1$ is an eigenfunction of the operator H_0 with $\omega = 0$ (this fact was first used for investigating the stability of a flame front in [25]). For one ionization or recombination wave $dS_0/d\xi$ is the eigenfunction $\Theta_0^{(0)}$ (we do not consider the boundary layer on the insulators), and therefore one ionization (recombination) wave is stable with respect to three-dimensional perturbations. For a soliton $dS_0/d\xi$ coincides with the eigenfunction $\Theta_1^{(0)}$, and for sufficiently small k_y the soliton is unstable $(\omega_0^{(0)} < 0)$.

The method considered above cannot be used to investigate the stability of a layered wave (or several layered waves) and several solitons, because of the structure of the spectrum of the operator H_0. Let l_c be the width of the layered wave (the distance between ionization and recombination waves).

In the case $l_c \to \infty$ the spectrum of the layered wave coincides with the spectrum of one ionization or recombination wave (the function $V(\xi)$ has one potential well, Fig. 4a), the distance between the eigenvalues ~ 1, and these eigenvalues are doubly degenerate. For finite values of l_c the spectrum of H_0 for the layered wave is obtained from the foregoing by breaking up each eigenvalue into two values lying close to each other which correspond to the symmetric and antisymmetric eigenfunctions (the layered wave is situated symmetrically in the channel, Fig. 4b). The distance between these eigenvalues is transcendentally small, and for ω_0 we can obtain [25]

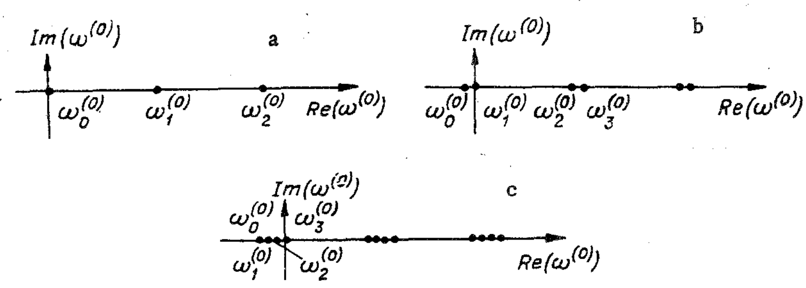

Fig. 4

$$\omega_0^{(0)} = -\sqrt{V(S_{03})}\exp\left(-\sqrt{V(S_{03})}\frac{l_c}{\Lambda}\right).$$

Since $A_{ml} \sim (\omega_m^{(0)} - \omega_l^{(0)})^{-1}$, it follows that among the eigenvalues ω there will always be some such $A_{ml} \sim O\cdot(\exp(\Lambda^{-1}))$, from which it follows that (2.7) is invalid. In the case of two layered waves the eigenvalues corresponding to $l_c \to \infty$ are divided into four levels lying close to each other (Fig. 4c). In this case there will also be transcendentally large coefficients A_{ml}.

Let us consider the construction of the first approximation for the layered wave. The appearance of transcendentally large quantities in the expansion (2.11) is due to the degeneracy of the eigenvalues, and the multiplicity of degeneracy for each eigenvalue is two. Since the operator H_0 is linear, every linear combination of two eigenfunctions corresponding to a degenerate level is also an eigenfunction of this level. Suppose that to the eigenvalue $\omega_m^{(0)}$ there correspond two eigenfunctions $\Theta_{m1}^{(0)}$ and $\Theta_{m2}^{(0)}$. We shall try to find the eigenfunction $\widehat{\Theta}_m^{(0)}$ of the degenerate value in the form

$$\widehat{\Theta}_m^{(0)} = A_1\Theta_{m1}^{(0)} + A_2\Theta_{m2}^{(0)}. \tag{2.14}$$

Substituting (2.14) into (2.10), multiplying the second equation first by $\Theta_{m1}^{(0)}$, and then by $\Theta_{m2}^{(0)}$, and integrating with respect to ξ from $-\Lambda^{-1}$ to Λ^{-1}, taking account of (2.9), we obtain a system for determining the values of A_1, A_2, and $\omega_m^{(1)}$:

$$\begin{aligned} A_1(\gamma_{m1m1} - \omega_m^{(1)}) + A_2\gamma_{m2m1} &= 0,\\ A_1\gamma_{m1m2} + A_2(\gamma_{m2m2} - \omega_m^{(1)}) &= 0, \end{aligned} \tag{2.15}$$

where $\gamma_{m1m2} = -2k_y^2E_{y0}^2\alpha_{m1m2} + ik_yE_{y0}\beta_{m1m2}$.

The condition for the solvability of the system (2.15) determines the value of $\omega_m^{(1)}$:

$$[\omega_m^{(1)}]_{1,2} = \frac{\gamma_{m1m1} + \gamma_{m2m2}}{2} \pm \sqrt{\frac{1}{4}(\gamma_{m1m1} - \gamma_{m2m2})^2 + \gamma_{m1m2}\gamma_{m2m1}}. \tag{2.16}$$

Let us estimate the order of the terms in Eq. (2.16). The functions $\Theta_{m1}^{(0)}$ and $\Theta_{m2}^{(0)}$ for the low levels are nonzero only in the regions of the potential wells, and in the other regions they are transcendentally small (the functions $\Theta_0^{(0)}$ and $\Theta_1^{(0)}$, corresponding to the degenerate level $\{\omega_0^{(0)}, \omega_1^{(0)}\}$, are represented by dashed curves in Fig. 3a, b, respectively). The function $G(\xi, \eta)$ depends on ξ and η only in terms of the combinations $\Lambda\xi$ and $\Lambda\eta$, and therefore for the quantity γ_{minj} we can obtain the following estimates:

$$\begin{aligned} \gamma_{m1m1} \approx \gamma_{m2m2} &\sim \mathrm{const}\,(G_1 + G_2) + O\left(\frac{1}{\Lambda}\exp\left(-\frac{1}{\Lambda}\right)\right),\\ \gamma_{m1m1} - \gamma_{m2m2} \sim \gamma_{m1m2} &\sim \gamma_{m2m1} \sim O\left(\frac{1}{\Lambda}\exp\left(-\frac{1}{\Lambda}\right)\right), \end{aligned} \tag{2.17}$$

where $G_1 = G(\xi = \xi_1, \eta = \xi_1, k)$; $G_2 = G(\xi = \xi_2, \eta = \xi_2, k)$; ξ_1, ξ_2 are the coordinates of the ionization and recombination waves; since the operator H_1 is symmetric with respect to $\xi = 0$, it follows that $G_1 = G_2$. Making use of (2.17), we can obtain, to within a transcendentally small quantity,

$$\omega_{m1}^{(1)} = \omega_{m2}^{(1)} = \gamma_{m1m1}. \tag{2.18}$$

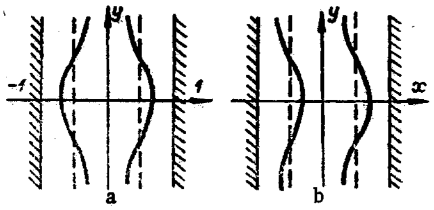

Fig. 5

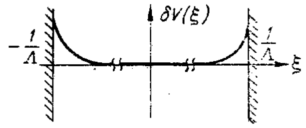

Fig. 6

Since a perturbation proportional to $dS_0/d\xi$, is simply a displacement of the wavefront [25], the perturbation $\Theta_0^{(0)}$ has the form of constrictions of the current column (Fig. 5a), and $\Theta_1^{(0)}$ is a deformation of the current column (Fig. 5b). The true perturbations of the layered wave $(\hat{\Theta}_0^{(0)}, \hat{\Theta}_1^{(0)})$ are linear combinations of these cases.

From (2.12), (2.18) it follows that the most unstable perturbations are those with $k_y = k_z = 0$. In this case the system (2.3) reduces to Eq. (2.8). For one layered wave or soliton the perturbation corresponding to $\omega_0^{(0)}$ is unstable, and the increment decreases as l_c increases; for the layered wave it is transcendentally small. This perturbation is a compression or elongation of the current column ($l_c = l_{c0} \pm \alpha \times \exp(-\omega_0^{(0)}t)$, $\alpha \ll 1$), and can be suppressed by the external circuit if that circuit has a sufficiently high resistance (in a specified-current regime this perturbation is, in general, inadmissible). For a soliton, since the zone of current inhomogeneity is small, the external circuit cannot substantially affect the perturbation increment [8]. Since a soliton is the limiting case of a layered wave (as $l_c \to 0$), for a specified resistance in the external circuit there exists a critical value of l_c above which the system becomes stable. Instability of a plane soliton leads to its breakdown into cylindrical solitons [8].

In cases with two layered waves $\omega_3^{(0)} = 0$ there are three types of perturbations for which the increment of instability is less than zero $(\omega_2^{(0)} < 0, \omega_1^{(0)} < 0, \omega_0^{(0)} < 0)$. A perturbation with $\omega = \omega_0^{(0)}$ is the simultaneous compression or expansion of the layered waves and leads to a change in the state of the external circuit, and therefore this perturbation can be suppressed by the external circuit.

In the development of a perturbation with $\omega = \omega_1^{(0)}$, the width of one wave decreases, while the width of the other increases. Since this perturbation is antisymmetric, it cannot be suppressed by the external circuit. A perturbation with $\omega = \omega_2^{(0)}$ is the motion of the waves either toward each other or away from each other. Even though this perturbation is symmetric, the external circuit has little effect on its development. From the foregoing it follows that the most unstable perturbation will be one with $\omega = \omega_1^{(0)}$, and this means that a soluution with two or more layered waves is unstable. Since the increment of this instability is transcendentally small, it appears that in gas-discharge experiments we can sometimes observe solutions with two layered waves [14].

When there is a boundary layer, we observe an additional potential $\delta V(\xi)$ (for the case when the temperature on the boundary is less than S_{01}, this is shown in Fig. 6). A correction to ω can be found by the method of perturbation [19]

$$\delta\omega_m = \int_{-\Lambda^{-1}}^{\Lambda^{-1}} [\Theta_m^{(0)}]^2 \, \delta V(\xi) \, d\xi.$$

Using the asymptotic form of $\Theta_m^{(0)}$, as $\xi \to \pm\Lambda^{-1}$, we obtain

$$\delta\omega_m \approx \int_{-\Lambda^{-1}}^{\Lambda^{-1}} \delta V(\xi) \, d\xi \exp\left\{-\sqrt{V(S_{01})}\,\frac{1+\frac{l_c}{2}}{\Lambda}\right\}.$$

We have the same order and sign for the additional term due to the difference from zero when $\xi = \pm\Lambda^{-1}$ of the quantity $a_1 d(dS_0/d\xi)/d\xi + a_2 dS_0/d\xi$. The quantity added to the frequency because of the boundary layer depends on the sign of $\delta V(\xi)$. In the case when the temperature at the wall is less than S_{01}, $\delta V(\xi)$ and $\delta\omega > 0$. Since the negative value for the layered wave is of the same order of magnitude as $\delta\omega_m$, the boundary layer may shift the negative eigenvalue $\omega_0^{(0)}$ into the stability region. If $\delta V(\xi) < 0$, the boundary layer leads to an additional instability in the inhomogeneous solutions.

LITERATURE CITED

1. I. Ya. Shipuk, "Ionization instability in a disk channel of a nonequilibrium MHD generator," in: The Magnetohydrodynamic Method of Energy Generation [in Russian], Énergiya, Moscow (1968).
2. S. B. Pashkin, "Hydrodynamic instability of plasma with current in a transverse field," Teplofiz. Vys. Temp., 9, No. 1 (1971).
3. N. A. Kruzhilin, "The development of overheat instability and acoustic instability in a plasma with molecular admixtures," Teplofiz. Vys. Temp., 11, No. 1 (1973).
4. A. M. Dykhne, "Theory of one-dimensional contraction of arcs," in: Some Problems in the Investigation of Gas-Discharge Plasma and the Generation of Strong Magnetic Fields [in Russian], Nauka, Leningrad (1970).
5. V. M. Atrazhev and I. T. Yakubov, "Contraction of a strongly nonequilibrium plasma with current," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1975).
6. V. D. Pis'mennyi and A. G. Rakhimov, "Instability of a strong gas discharge," Dokl. Akad. Nauk SSSR, 196 (1971).
7. A. F. Vitshas, A. M. Dykhne, V. G. Naumov, and V. N. Panchenko, "Investigation of a quasi-one-dimensional model of a contraction of a gas discharge," Teplofiz. Vys. Temp., 9, No. 2 (1971).
8. A. F. Volkov and Sh. M. Kogan, "The occurrence of nonuniform current distribution in semiconductors with negative differential conductivity," Zh. Eksp. Teor. Fiz. 52, No. 6 (1967).
9. Sh. M. Kogan, "A current column in a semiconductor with S-shaped current–voltage characteristic," Zh. Eksp. Teor. Fiz., 54, No. 4 (1968).
10. A. F. Volkov and Sh. M. Kogan, "Physical phenomena in semiconductors with negative differential conductivity," Usp. Fiz. Nauk, 96, No. 2 (1968).
11. S. V. Pashkin, "Contraction of a plasma with current," Teplofiz. Vys. Temp., 10, No. 6 (1972).
12. V. Yu. Baranov and K. N. Ul'yanov, "Contraction of a positive column," Zh. Tekh. Fiz., 39 (1969).
13. V. Yu. Baranov and K. N. Ul'yanov, "Contraction of a positive column," Pisma Zh. Eksp. Teor. Fiz., 6, No. 5 (1967).
14. A. F. Vitshas, V. S. Golubev, and M. M. Malikov, "The value of the normal current density in a contracted plasma," in: Some Problems in the Investigation of a Gas-Discharge Plasma and the Generation of Strong Magnetic Fields [in Russian], Nauka, Leningrad (1970).
15. B. B. Kadomtsev, "Hydromagnetic stability of a plasma," in: Problems in Plasma Theory [in Russian], No. 2, Gosatomizdat, Moscow (1963).
16. Yu. V. Sanochkin, "Dissipative instability of a nonisothermal current in an electrical conductor between parallel plates in a transverse magnetic field," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1967).
17. S. P. Kurdyumov, Yu. P. Popov, and A. A. Samarskii, "Nonlinear effects of the formation of structures in radiation magnetohydrodynamics," in: Heat and Mass Exchange [in Russian], Vol. 8, Minsk (1972).
18. B. W. Knight and G. A. Peterson, "Theory of the Gunn effect," Phys. Rev., 155, No. 2 (1967).
19. O. A. Sinkevich, "Stability of inhomogeneous stationary states in a nonequilibrium magnetized plasma," Prikl. Mat. Mekh., 41, No. 6 (1977).
20. V. I. Artemov and O. A. Sinkevich, "Construction of plane stationary solutions of the equations of a nonequilibrium magnetized plasma," Prikl. Mat. Mekh., 40, No. 5 (1976).
21. V. M. Zubtsov and O. A. Sinkevich, "Development of two-dimensional and three-dimensional perturbations in the case of ionization instability in a channel with nonconductive walls," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 3 (1976).
22. O. A. Sinkevich, "The nature of the stability loss in a nonequilibrium magnetized plasma," Prikl. Mat. Mekh., 38, No. 4 (1974).
23. R. Courant and D. Hilbert, Methods of Mathematical Physics, Wiley (1962).
24. Ali-Hasan Nayfeh, Perturbation Methods, Wiley (1973).
25. G. I. Barenblat and Ya. B. Zel'dovich, "Stability of the propagation of a flame," Prikl. Mat. Mekh., 21, No. 6 (1957).